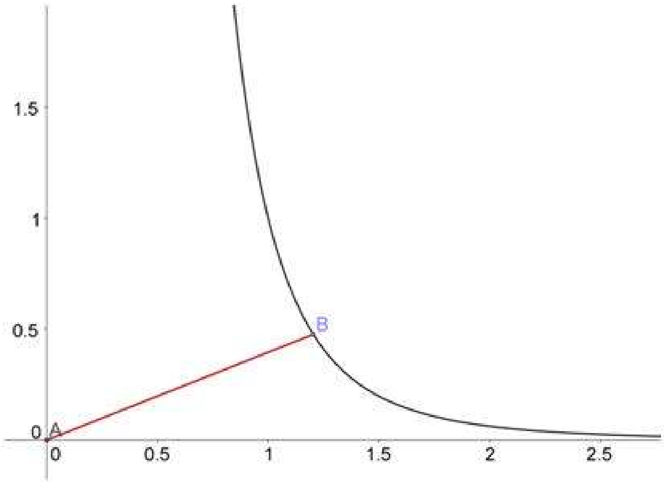
A minimum problem

Find the shortest distance from the origin O to the curve $y = \frac{1}{x^4}$ (where x > 0).

Use the following methods:

(1) Calculus,

(2) A.M.≥ G.M.



(1) Calculus:

Method 1

$$s = AB = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{1}{x^4}\right)^2} = \sqrt{x^2 + \frac{1}{x^8}}$$

$$s^2 = x^2 + \frac{1}{x^8}$$

$$2s\frac{ds}{dx} = 2x - \frac{8}{x^9} \Rightarrow \frac{ds}{dx} = \frac{1}{s}\left(\frac{x^{10}-4}{x^9}\right)$$

For critical points, $\frac{ds}{dx} = 0$.

$$x^{10} - 4 = 0 \Longrightarrow \quad x = \sqrt[10]{4} = \sqrt[5]{2} \approx 1.148698354997$$

For $0 < x < \sqrt[5]{2}$, $\frac{ds}{dx} < 0$. and for $x > \sqrt[5]{2}$, $\frac{ds}{dx} < 0$.

Therefore y is a minimum when $x = \sqrt[5]{2}$.

When $x = \sqrt[5]{2}$,

$$s = \sqrt{\left(\sqrt[5]{2}\right)^2 + \frac{1}{\left(\sqrt[5]{2}\right)^8}} = \sqrt{\frac{\left(\sqrt[5]{2}\right)^{10} + 1}{\left(\sqrt[5]{2}\right)^8}} = \sqrt{\frac{4+1}{\sqrt[5]{256}}} = \sqrt{\frac{5}{\sqrt[5]{256}}} = \sqrt[10]{\frac{3125}{256}} \approx 1.2842838037078$$

Method 2

Construct a circle centre origin and radius r :

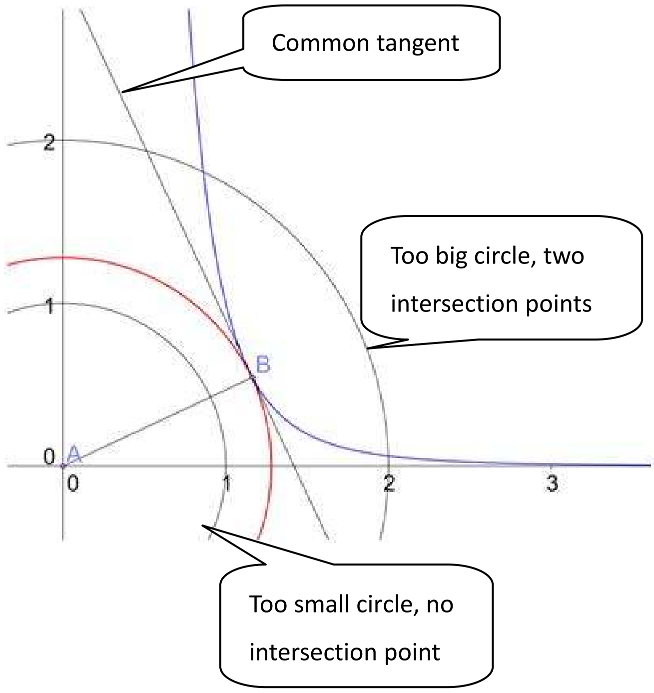
$$x^2 + y^2 = r^2 \qquad .... \quad (1)$$

In order to find the shortest r, we like to have this circle **touches** the given curve:

$$y = \frac{1}{x^4} \qquad .... \quad (2)$$

Then (1) and (2) should have a common tangent

at the point of contact .



Differentiate (1) and (2),

$$\begin{cases} 2x + 2y\dfrac{dy}{dx} = 0 \quad \Longrightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{y} \\ \qquad\qquad \dfrac{dy}{dx} = -\dfrac{4}{x^5} \end{cases}$$

$$\therefore \quad -\frac{x}{y} = -\frac{4}{x^5} \implies y = \frac{x^6}{4} \qquad .... \quad (3)$$

By (2), $\quad \frac{1}{x^4} = \frac{x^6}{4} \implies x^{10} = 4 \implies x = \sqrt[10]{4} = \sqrt[5]{2} \approx 1.148698354997$

By (3), $\quad y = \frac{\left(\sqrt[5]{2}\right)^6}{4}$

By (1), $\quad r^2 = x^2 + y^2 = \left(\sqrt[5]{2}\right)^2 + \left[\frac{\left(\sqrt[5]{2}\right)^6}{4}\right]^2 \approx 1.6493848884661$

$$r \approx 1.2842838037078$$

(2) A.M.≥ G.M. :

$$s = AB = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{1}{x^4}\right)^2} = \sqrt{x^2 + \frac{1}{x^8}}$$

$$= \sqrt{\frac{x^2}{4} + \frac{x^2}{4} + \frac{x^2}{4} + \frac{x^2}{4} + \frac{1}{x^8}} \le \sqrt{5\sqrt[5]{\left(\frac{x^2}{4}\right)\left(\frac{x^2}{4}\right)\left(\frac{x^2}{4}\right)\left(\frac{x^2}{4}\right)\left(\frac{1}{x^8}\right)}} \qquad \text{(A.M.} \ge \text{G.M.)}$$

The point is to get rid of all x !

$$= \sqrt{5\sqrt[5]{\frac{1}{256}}} = \sqrt[10]{\frac{3125}{256}} \approx 1.2842838037078$$

where equality holds if and only if

$$\frac{x^2}{4} = \frac{1}{x^8} \iff x^{10} = 4 \iff x = \sqrt[10]{4} = \sqrt[5]{2} \approx 1.148698354997$$

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